

# AUTOMORPHISMS OF THE LIE ALGEBRA OF VECTOR FIELDS ON AFFINE $n$ -SPACE

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**ABSTRACT.** We show that every Lie algebra automorphisms of the vector fields  $\text{Vec}(\mathbb{A}^n)$  of affine  $n$ -space  $\mathbb{A}^n$ , of the vector fields  $\text{Vec}^c(\mathbb{A}^n)$  with constant divergence, and of the vector fields  $\text{Vec}^0(\mathbb{A}^n)$  with divergence zero is induced by an automorphism of  $\mathbb{A}^n$ . This generalizes results of the second author obtained in dimension 2, see [Reg13]. The case of  $\text{Vec}(\mathbb{A}^n)$  is due to BAVULA [Bav13].

As an immediate consequence, we get the following result due to KULIKOV [Kul92]. If every injective endomorphism of the Lie algebra  $\text{Vec}(\mathbb{A}^n)$  is an automorphism, then the Jacobian Conjecture holds in dimension  $n$ .

## 1. INTRODUCTION

Let  $K$  be an algebraically closed field of characteristic zero. Denote by  $\text{Vec}(\mathbb{A}^n)$  the Lie algebra of polynomial vector fields on affine  $n$ -space  $\mathbb{A}^n = K^n$ . We have the standard identifications

$$\text{Vec}(\mathbb{A}^n) = \text{Der}(K[x_1, \dots, x_n]) = \left\{ \sum_i f_i \frac{\partial}{\partial x_i} \mid f_i \in K[x_1, \dots, x_n] \right\}.$$

The group  $\text{Aut}(\mathbb{A}^n)$  of polynomial automorphisms of  $\mathbb{A}^n$  acts on  $\text{Vec}(\mathbb{A}^n)$  in the usual way. For  $\varphi \in \text{Aut}(\mathbb{A}^n)$  and  $\delta \in \text{Vec}(\mathbb{A}^n)$  we define

$$\text{Ad}(\varphi)\delta := \varphi^{*-1} \circ \delta \circ \varphi^*$$

where we consider  $\delta$  as a derivation  $\delta: K[x_1, \dots, x_n] \rightarrow K[x_1, \dots, x_n]$  and where  $\varphi^*: K[x_1, \dots, x_n] \rightarrow K[x_1, \dots, x_n]$ ,  $f \mapsto f \circ \varphi$ , is the co-morphism of  $\varphi$ . It is shown in [Bav13] that  $\text{Ad}: \text{Aut}(\mathbb{A}^n) \rightarrow \text{Aut}_{\text{Lie}}(\text{Vec}(\mathbb{A}^n))$  is an isomorphism. We will give a short proof in section 3.

Recall that the *divergence* of a vector field  $\delta = \sum_i f_i \frac{\partial}{\partial x_i}$  is defined by  $\text{Div } \delta := \sum_i \frac{\partial f_i}{\partial x_i}$ . This allows to define the following subspaces of  $\text{Vec}(\mathbb{A}^n)$ :

$$\text{Vec}^0(\mathbb{A}^n) := \{\delta \in \text{Vec}(\mathbb{A}^n) \mid \text{Div } \delta = 0\} \subset \text{Vec}^c(\mathbb{A}^n) := \{\delta \in \text{Vec}(\mathbb{A}^n) \mid \text{Div } \delta \in K\},$$

which are Lie subalgebras, because  $\text{Div}[\delta, \eta] = \delta(\text{Div } \eta) - \eta(\text{Div } \delta)$ . We have

$$\text{Vec}^c(\mathbb{A}^n) = \text{Vec}^0(\mathbb{A}^n) \oplus KE \text{ where } E := \sum_i x_i \frac{\partial}{\partial x_i} \text{ is the Euler field.}$$

*Remark 1.1.* The group  $\text{Aut}(\mathbb{A}^n)$  has the structure of an *ind-group*, i.e. an *infinite dimensional algebraic group* in the sense of SHAFAREVICH (see [Sha66, Sha81], cf. [Kum02]). One can show that its Lie algebra is canonically isomorphic to  $\text{Vec}^c(\mathbb{A}^n)$ . This is one of the reasons for studying this Lie algebra and its properties.

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The aim of this note is to prove the following result about the automorphism groups of these Lie algebras.

**Main Theorem.** *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

*Remark 1.2.* It is easy to see that the theorem holds for any field  $K$  of characteristic zero. In fact, all the homomorphisms are defined over  $\mathbb{Q}$ , and are equivariant with respect to the obvious actions of the Galois group  $\Gamma = \mathrm{Gal}(\bar{K}/K)$ .

As a consequence, we will get the following result which is due to KULIKOV, see Corollary 4.4.

**Corollary.** *If every injective endomorphism of the Lie algebra  $\mathrm{Vec}(\mathbb{A}^n)$  is an automorphism, then the Jacobian Conjecture holds in dimension  $n$ .*

*Remark 1.3.* It was proved by BELOV-KANEL and YU that every automorphism of  $\mathrm{Aut}(\mathbb{A}^n)$  as an ind-group is inner (see [BKY12]). Using the main results here one can give a short of this and extend it to the closed subgroup  $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$  of automorphism with Jacobian determinant equal to 1, see [KZ13].

We add here a lemma which will be used later on. The first statement is in [Sha81, Lemma 3], and from that the second follows immediately.

**Lemma 1.4.**  *$\mathrm{Vec}^0(\mathbb{A}^n)$  is a simple Lie algebra, and  $\mathrm{Vec}^0(\mathbb{A}^n) = [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)]$ .*

## 2. GROUP ACTIONS AND VECTOR FIELDS

If an algebraic group  $G$  acts on an affine variety  $X$  we obtain a canonical map  $\mathrm{Lie} G \rightarrow \mathrm{Vec}(X)$  in the usual way (cf. [Kra11, II.4.4]). For every  $A \in \mathrm{Lie} G$  the associated vector field  $\xi_A$  on  $X$  is defined by

$$(1) \quad (\xi_A)_x := d\mu_x(A) \text{ for } x \in X$$

where  $\mu_x: G \rightarrow X$ ,  $g \mapsto gx$ , is the orbit map in  $x \in X$ . It is well-known that the linear map  $A \mapsto \xi_A$  is a anti-homomorphism of Lie algebras, and that the kernel is equal to the Lie algebra of the kernel of the action  $G \rightarrow \mathrm{Aut}(X)$ . In particular, for any algebraic subgroup  $G \subset \mathrm{Aut}(\mathbb{A}^n)$  we have an injection  $\mathrm{Lie} G \rightarrow \mathrm{Vec}(\mathbb{A}^n)$ . We will denote the image by  $L(G)$ . Let us point out that a connected  $G \subset \mathrm{Aut}(\mathbb{A}^n)$  is determined by  $L(G)$ , i.e. if  $L(G) = L(H)$  for algebraic subgroups  $G, H \subset \mathrm{Aut}(\mathbb{A}^n)$ , then  $G^0 = H^0$ .

Recall that the vector field  $\delta \in \mathrm{Vec}(\mathbb{A}^n)$  is called *locally nilpotent* if the action of  $\delta$  on  $K[x_1, \dots, x_n]$  is locally nilpotent, i.e., for any  $f \in K[x_1, \dots, x_n]$  we have  $\delta^m(f) = 0$  if  $m$  is large enough. Every such  $\delta$  defines an action of the additive group  $K^+$  on  $\mathbb{A}^n$  such that  $\delta = \xi_1$  where  $1 \in K = \mathrm{Lie} K^+$  (see (1) above).

**Lemma 2.1.** *Let  $\mathfrak{u} \subset \mathrm{Vec}(\mathbb{A}^n)$  be a finite dimensional commutative Lie subalgebra consisting of locally nilpotent vector fields. Then there is a commutative unipotent algebraic subgroup  $U \subset \mathrm{Aut}(\mathbb{A}^n)$  such that  $L(U) = \mathfrak{u}$ . If  $\mathrm{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathfrak{u}) = \mathfrak{u}$ , then  $U$  acts transitively on  $\mathbb{A}^n$ .*

*Proof.* It is clear that  $\mathfrak{u} = L(U)$  for a commutative unipotent subgroup  $U \subset \mathrm{Aut}(\mathbb{A}^n)$ . In fact, choose a basis  $(\delta_1, \dots, \delta_m)$  of  $\mathfrak{u}$  and consider the corresponding actions  $\rho_i: K^+ \rightarrow \mathrm{Aut}(\mathbb{A}^n)$ . Since the associated vector fields  $\delta_i$  commute, the

same holds for the actions  $\rho_i$ , so that we get an action of  $(K^+)^m$ . It follows that the image  $U \subset \text{Aut}(\mathbb{A}^n)$  is a commutative unipotent subgroup with  $L(U) = \mathfrak{u}$ .

Assume that the action of  $U$  is not transitive. Then all orbits have dimension  $< n$ , because orbits under unipotent groups are closed. But then there is a non-constant  $U$ -invariant function  $f \in K[x_1, \dots, x_n]$ . This implies that for every  $\delta \in \mathfrak{u}$  the vector field  $f\delta$  commutes with  $\mathfrak{u}$  and thus belongs to  $\mathfrak{cent}_{\text{Vec}(\mathbb{A}^n)}(\mathfrak{u})$ , contradicting the assumption.  $\square$

Any  $\delta \in \text{Vec}(\mathbb{A}^n)$  acts on the functions  $K[x_1, \dots, x_n]$  as a derivation, and on the Lie algebra  $\text{Vec}(\mathbb{A}^n)$  by the adjoint action,  $\text{ad}(\delta)\mu := [\delta, \mu]$ . These two actions are related as shown in the following lemma whose proof is obvious.

**Lemma 2.2.** *Let  $\delta, \mu \in \text{Vec}(\mathbb{A}^n)$  be two commuting vector fields. Then*

$$\text{ad}(\delta)(f\mu) = \delta(f)\mu.$$

*In particular, if  $\text{ad}(\delta)$  is locally nilpotent on  $\text{Vec}(\mathbb{A}^n)$ , then  $\delta$  is locally nilpotent.*

### 3. PROOF OF THE MAIN THEOREM, PART I

We first give a proof of the following result due to BAVULA [Bav13].

**Theorem 3.1.** *The canonical map  $\text{Ad}: \text{Aut}(\mathbb{A}^n) \rightarrow \text{Aut}_{\text{Lie}}(\text{Vec}(\mathbb{A}^n))$  is an isomorphism.*

Denote by  $\text{Aff}_n \subset \text{Aut}(\mathbb{A}^n)$  the closed subgroup of affine transformations and by  $S = (K^+)^n \subset \text{Aff}_n$  the subgroup of translations. Then

$$(2) \quad L(\text{Aff}_n) = \langle x_i \partial_{x_j}, \partial_{x_k} \mid 1 \leq i, j, k \leq n \rangle \supset L(S) = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle.$$

where  $\partial_{x_j} := \frac{\partial}{\partial x_j}$ . Put  $\mathfrak{aff}_n := \text{Lie Aff}_n$  and  $\mathfrak{saff}_n := [\mathfrak{aff}_n, \mathfrak{aff}_n]$ . We have  $\mathfrak{saff}_n := \text{Lie SAff}_n$  where  $\text{SAff}_n := (\text{Aff}_n, \text{Aff}_n) \subset \text{Aff}_n$  is the commutator subgroup, i.e. the closed subgroup of those affine transformations  $x \mapsto gx + b$  where  $g \in \text{SL}_n$ . The next lemma is certainly known. For the convenience of the reader we indicated a short proof.

**Lemma 3.2.** *The canonical homomorphisms*

$$\text{Aff}_n \xrightarrow[\simeq]{\text{Ad}} \text{Aut}_{\text{Lie}}(\mathfrak{aff}_n) \xrightarrow[\simeq]{\text{res}} \text{Aut}_{\text{Lie}}(\mathfrak{saff}_n)$$

*are isomorphisms.*

*Proof.* It is clear that the two homomorphisms  $\text{Ad}: \text{Aff}_n \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{aff}_n)$  and  $\text{res}: \text{Aut}_{\text{Lie}}(\mathfrak{aff}_n) \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{saff}_n)$  are both injective. Thus it suffices to show that the composition  $\text{res} \circ \text{Ad}$  is surjective.

We write the elements of  $\text{Aff}_n$  in the form  $(v, g)$  with  $v \in S = (K^+)^n$ ,  $g \in \text{GL}_n$  where  $(v, g)x = gx + v$  for  $x \in \mathbb{A}^n$ . It follows that  $(v, g)(w, h) = (v + gw, gh)$ . Similarly,  $(a, A) \in \mathfrak{aff}_n$  means that  $a \in \mathfrak{s} = \text{Lie } S = (K)^n$ ,  $A \in \mathfrak{gl}_n$ , and  $(a, A)x = Ax + a$ . For the adjoint representation of  $g \in \text{GL}_n$  and of  $v \in S$  on  $\mathfrak{aff}_n$  we find

$$(3) \quad \text{Ad}(g)(a, A) = (ga, gAg^{-1}) \quad \text{and} \quad \text{Ad}(v)(a, A) = (a - Av, A),$$

and thus, for  $(b, B) \in \mathfrak{aff}_n$ ,

$$(4) \quad \text{ad}(B)(a, A) = (Ba, [B, A]) \quad \text{and} \quad \text{ad}(b)(a, A) = (a - Ab, A).$$

Now let  $\theta$  be an automorphism of the Lie algebra  $\mathfrak{sa}\mathfrak{ff}_n$ . Then  $\theta(\mathfrak{s}) = \mathfrak{s}$  since  $\mathfrak{s}$  is the solvable radical of  $\mathfrak{sa}\mathfrak{ff}_n$ . Since  $g := \theta|_{\mathfrak{s}} \in \mathrm{GL}_n$ , we can replace  $\theta$  by  $\mathrm{Ad}(g^{-1}) \circ \theta$  and thus assume, by (3), that  $\theta$  is the identity on  $\mathfrak{t}$ . This implies that  $\theta(a, A) = (a + \ell(A), \bar{\theta}(A))$  where  $\ell: \mathfrak{sl}_n \rightarrow \mathfrak{s}$  is a linear map and  $\bar{\theta}: \mathfrak{sl}_n \xrightarrow{\sim} \mathfrak{sl}_n$  is a Lie algebra automorphism.

From (4) we get  $\mathrm{ad}(b, B)(a, 0) = \mathrm{ad}(B)(a, 0) = (Ba, 0)$  for all  $a \in \mathfrak{s}$ , hence

$$\begin{aligned} (Ba, 0) &= \theta(Ba, 0) = \theta(\mathrm{ad}(B)(a, 0)) = \\ &= \mathrm{ad}(\theta(B))(a, 0) = \mathrm{ad}(\bar{\theta}(B))(a, 0) = (\bar{\theta}(B)a, 0). \end{aligned}$$

Thus  $\bar{\theta}(B) = B$ , i.e.  $\theta(a, A) = (a + \ell(A), A)$ . Now an easy calculation shows that  $\ell([A, B]) = A\ell(B) - B\ell(A)$ . This means that  $\ell$  is a cocycle of  $\mathfrak{sl}_n$ . Since  $\mathfrak{sl}_n$  is semisimple,  $\ell$  is a coboundary and thus  $\ell(A) = Av$  for a suitable  $v \in K^n$ . In view of (4) this implies that  $\theta = \mathrm{Ad}(-v)$ , and the claim follows.  $\square$

*Proof of Theorem 3.1.* It is clear that the homomorphism

$$\mathrm{Ad}: \mathrm{Aut}(\mathbb{A}^n) \rightarrow \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$$

is injective. So let  $\theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$  be an arbitrary automorphism.

We have seen above that  $L(S) = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \subset \mathrm{Vec}(\mathbb{A}^n)$  where  $S \subset \mathrm{Aff}_n$  is the subgroup of translations. Since  $\mathrm{Vec}(\mathbb{A}^n) = K[x_1, \dots, x_n]L(S)$  we get from Lemma 2.2 that the adjoint action of any  $\delta \in L(S)$  on  $\mathrm{Vec}(\mathbb{A}^n)$  is locally nilpotent, and the same holds for any element from  $\mathfrak{u} := \theta(L(S))$ . This implies, by Lemma 2.1, that  $\mathfrak{u} = L(U)$  for a commutative unipotent subgroup  $U$  of dimension  $n$ . Moreover,  $\mathrm{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(L(S)) = L(S)$ , hence  $\mathrm{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathfrak{u}) = \mathfrak{u}$  which implies, again by Lemma 2.1, that  $U$  acts transitively on  $\mathbb{A}^n$ . Thus every orbit map  $U \rightarrow \mathbb{A}^n$  is an isomorphism. It follows that there is an automorphism  $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$  such that  $\varphi U \varphi^{-1} = S$ . In fact, fix a group isomorphism  $\varphi: U \xrightarrow{\sim} S$  and take the orbit maps  $\mu_S: S \xrightarrow{\sim} \mathbb{A}^n$  and  $\mu_U: U \xrightarrow{\sim} \mathbb{A}^n$  at the origin  $0 \in \mathbb{A}^n$ . Then  $\varphi := \mu_S \circ \varphi \circ \mu_U^{-1}$  has the property that  $\varphi^{-1}u\varphi = \varphi(u)$  for all  $u \in U$ .

It follows that the automorphism  $\theta' := \mathrm{Ad}(\varphi) \circ \theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$  sends  $L(S)$  isomorphically onto itself. Now the relations  $[\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_k}] = \delta_{ij} \frac{\partial}{\partial x_k}$  imply that  $\theta'(L(\mathrm{Aff}_n)) = L(\mathrm{Aff}_n)$ , and from Lemma 3.2 we obtain an affine automorphism  $\tau \in \mathrm{Aff}_n$  such that  $\mathrm{Ad}(\tau) \circ \theta'$  is the identity on  $L(\mathrm{Aff}_n)$ . Hence, by the following lemma,  $\mathrm{Ad}(\tau) \circ \theta' = \mathrm{id}$ , and the claim follows.  $\square$

**Lemma 3.3.** *Let  $\theta$  be an injective endomorphism of one of the Lie algebras  $\mathrm{Vec}(\mathbb{A}^n)$ ,  $\mathrm{Vec}^c(\mathbb{A}^n)$  or  $\mathrm{Vec}^0(\mathbb{A}^n)$ . If  $\theta$  is the identity on  $L(\mathrm{SL}_n)$ , then  $\theta = \mathrm{Ad}(\lambda E)$  for some  $\lambda \in K^*$ .*

*Proof.* We consider the action of  $\mathrm{GL}_n$  on  $\mathrm{Vec}(\mathbb{A}^n)$ . Denote by  $\mathrm{Vec}(\mathbb{A}^n)_d$  the homogeneous vector fields of degree  $d$ , i.e.

$$\mathrm{Vec}(\mathbb{A}^n)_d := \bigoplus_i K[x_1, \dots, x_n]_{d+1} \partial_{x_i} \simeq K[x_1, \dots, x_n]_{d+1} \otimes K^n.$$

Note that  $\lambda E \in \mathrm{GL}_n$  acts by scalar multiplication with  $\lambda^{-d}$  on  $\mathrm{Vec}(\mathbb{A}^n)_d$ . We have split exact sequences of  $\mathrm{GL}_n$ -modules

$$0 \longrightarrow \mathrm{Vec}^0(\mathbb{A}^n)_d \longrightarrow \mathrm{Vec}(\mathbb{A}^n)_d \xrightarrow{\mathrm{Div}} K[x_1, \dots, x_n]_{d+1} \longrightarrow 0$$

where all  $\mathrm{SL}_n$ -modules  $\mathrm{Vec}^0(\mathbb{A}^n)_d$  and  $K[x_1, \dots, x_n]_{d+1}$  are simple and pairwise non-isomorphic (see PIERI's formula [Pro07, Chap. 9, section 10.2]).

Now let  $\theta$  be an automorphism of  $\text{Vec}(\mathbb{A}^n)$ . If  $\theta$  is the identity on  $L(\text{SL}_n)$ , then  $\theta$  is  $\text{SL}_n$ -equivariant and thus acts with a scalar  $\lambda_d$  on  $\text{Vec}^0(\mathbb{A}^n)_d$  and with a scalar  $\mu_d$  on  $K[x_1, \dots, x_n]_{d+1}$ . The relation

$$[x_j^{e+1} \partial_{x_i}, x_i^{d+1} \partial_{x_j}] = (d+1)x_i^d x_j^{e+1} \partial_{x_j} - (e+1)x_i^{d+1} x_j^e \partial_{x_j}$$

shows that  $\lambda_e \lambda_d = \lambda_{e+d}$ , hence  $\lambda_d = \lambda^d$  for a suitable  $\lambda \in K^*$ . It follows that the composition  $\theta' := \text{Ad}(\lambda E) \circ \theta$  is the identity on  $\text{Vec}^0(\mathbb{A}^n)$ . Now we use the Euler field  $\partial_E$  and the relation  $[\partial_E, \delta] = d \cdot \delta$  for  $\delta \in \text{Vec}(\mathbb{A}^n)_d$  to see that  $\theta'$  is the identity everywhere. This proves the claim for  $\text{Vec}(\mathbb{A}^n)$ . The two other cases are similar.  $\square$

#### 4. ÉTALE MORPHISMS AND VECTOR FIELDS

In the first section we defined the action of  $\text{Aut}(\mathbb{A}^n)$  on the vector fields  $\text{Vec}(\mathbb{A}^n)$  by the usual formula  $\text{Ad}(\varphi)\delta := \varphi^{*-1} \circ \delta \circ \varphi^*$ . In more geometric terms, considering  $\delta$  as a section of the tangent bundle  $T\mathbb{A}^n = \mathbb{A}^n \times \mathbb{C}^n \rightarrow \mathbb{A}^n$ , one defines the pull-back of  $\delta$  by

$$(5) \quad \varphi^*(\delta) := (d\varphi)^{-1} \circ \delta \circ \varphi, \text{ i.e., } \varphi^*(\delta)_a = (d\varphi_a)^{-1}(\delta_{\varphi(a)}) \text{ for } a \in \mathbb{A}^n.$$

Clearly,  $\varphi^*(\delta) = \text{Ad}(\varphi^{-1})\delta$ . However, the second formula above shows that the pull-back  $\varphi^*(\delta)$  of a vector field is also defined for étale morphisms  $\varphi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ .

**Proposition 4.1.** *Let  $\varphi: \mathbb{A}^n \rightarrow \mathbb{A}^n$  be an étale morphism. For any vector field  $\delta \in \text{Vec}(\mathbb{A}^n)$  there is a uniquely defined vector field  $\varphi^*(\delta)$  such that*

$$(6) \quad d\varphi \circ \varphi^*(\delta) = \delta \circ \varphi.$$

*The map  $\varphi^*: \text{Vec}(\mathbb{A}^n) \rightarrow \text{Vec}(\mathbb{A}^n)$  is an injective homomorphism of Lie algebras. Moreover,  $(\eta \circ \varphi)^* = \varphi^* \circ \eta^*$ .*

*Proof.* For a vector field  $\xi: \mathbb{A}^n \rightarrow T\mathbb{A}^n$  and  $a \in \mathbb{A}^n$  we have  $(d\varphi \circ \xi)_a = \text{Jac}(\varphi)_a \cdot \xi_a$ . Thus, for a given  $\delta \in \text{Vec}(\mathbb{A}^n)$ , the equation  $(d\varphi \circ \tilde{\delta})_a = (\delta \circ \varphi)_a = \delta_{\varphi(a)}$  has the unique solution

$$\tilde{\delta}_a := (\text{Jac}(\varphi)_a)^{-1} \cdot \delta_{\varphi(a)}.$$

Since the Jacobian determinant  $\det(\text{Jac}(\varphi))$  is a non-zero constant, the inverse matrix  $\text{Jac}(\varphi)^{-1}$  has entries in  $K[x_1, \dots, x_n]$ . Therefore, the vector field  $\tilde{\delta}$  is polynomial. This proves the first claim of the proposition, and the others follow immediately.  $\square$

*Remark 4.2.* In the notation of the proposition above, let  $\varphi = (f_1, \dots, f_n)$  and put  $\delta = \partial_{x_i}$  in equation (6). Then

$$\partial_{x_i} = \sum_j \frac{\partial f_j}{\partial x_i} \varphi^*(\partial_{x_j}),$$

Applying both sides to  $f_k$ , we get  $\varphi^*(\partial_{x_j})(f_k) = \delta_{jk}$ .

**Proposition 4.3.** *Let  $\eta: \mathbb{A}^n \rightarrow \mathbb{A}^n$  be an étale morphism. If  $\eta^*: \text{Vec}(\mathbb{A}^n) \rightarrow \text{Vec}(\mathbb{A}^n)$  is an isomorphism, then  $\eta$  is an automorphism of  $\mathbb{A}^n$ .*

*Proof.* Theorem 3.1 implies that  $\eta^* = \text{Ad}(\varphi)$  for some automorphism  $\varphi$  of  $\mathbb{A}^n$ . Since  $\varphi^* = \text{Ad}(\varphi^{-1})$ , it follows that  $\psi := \varphi \circ \eta$  is étale and that  $\psi^*$  is the identity on  $\text{Vec}(\mathbb{A}^n)$ . We claim that this implies that  $\psi = \text{id}$  which will prove the proposition.

By definition, we have  $\delta_a = (\text{Jac}(\psi)_a)^{-1} \cdot \delta_{\psi(a)}$  for every vector field  $\delta$ . Putting  $\delta = \partial_{x_i}$  for  $i = 1, \dots, n$ , we get  $\text{Jac}(\psi)_a = E$  for all  $a \in \mathbb{A}^n$  which implies that  $\psi$  is a translation.  $\square$

As a corollary of the two propositions above, we get the following result which is due to KULIKOV [Kul92, Theorem 4].

**Corollary 4.4.** *If every injective endomorphism of the Lie algebra  $\text{Vec}(\mathbb{A}^n)$  is an automorphism, then the Jacobian Conjecture holds in dimension  $n$ .*

We finish this section by showing that if the divergence of a vector field is a constant, then it does not change under an étale morphism. More general, we have the following result.

**Proposition 4.5.** *Let  $\eta: \mathbb{A}^n \rightarrow \mathbb{A}^n$  be an étale morphism, and  $\delta$  a vector field. Then  $\text{Div } \eta^*(\delta) = \eta^*(\text{Div } \delta)$ . In particular,  $\delta \in \text{Vec}^c(\mathbb{A}^n)$  if and only if  $\eta^*(\delta) \in \text{Vec}^c(\mathbb{A}^n)$ , and in this case we have  $\text{Div } \eta^*(\delta) = \text{Div } \delta$ .*

*Proof.* Set  $\eta = (f_1, \dots, f_n)$ ,  $\delta = \sum_j h_j \frac{\partial}{\partial x_j}$  and  $\eta^*(\delta) = \sum_j \tilde{h}_j \frac{\partial}{\partial x_j}$ . Then

$$h_k(f_1, \dots, f_n) = \sum_i \tilde{h}_i \frac{\partial f_k}{\partial x_i} \text{ for } k = 1, \dots, n.$$

Applying  $\frac{\partial}{\partial x_j}$  to the left hand side we get the matrix

$$\left( \sum_i \frac{\partial h_k}{\partial x_i}(f_1, \dots, f_n) \frac{\partial f_i}{\partial x_j} \right)_{(k,j)} = H(f_1, \dots, f_n) \cdot \text{Jac}(\eta)$$

where  $H := \text{Jac}(h_1, \dots, h_n)$ . On the right hand side, we obtain similarly

$$\left( \sum_i \frac{\partial \tilde{h}_i}{\partial x_j} \frac{\partial f_k}{\partial x_i} + \sum_i \tilde{h}_i \frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{(k,j)} = \tilde{H} \cdot \text{Jac}(\eta) + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i} \text{Jac}(\eta)$$

Multiplying this matrix equation from the right with  $\text{Jac}(\eta)^{-1}$  we finally get

$$H(f_1, \dots, f_n) = \tilde{H} + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i} \text{Jac}(\eta) \cdot \text{Jac}(\eta)^{-1}$$

Now we take on both sides the traces. Using Lemma 4.6 below and the obvious equalities  $\text{Div } \delta = \text{tr } H$  and  $\text{Div } \tilde{\delta} = \text{tr } \tilde{H}$ , we finally get

$$\text{Div } \tilde{\delta} = (\text{Div } \delta)(f_1, \dots, f_n) = \eta^*(\text{Div } \delta).$$

The claim follows.  $\square$

**Lemma 4.6.** *Let  $A$  be an  $n \times n$  matrix whose entries  $a_{ij}(t)$  are differentiable function in  $t$ . Then*

$$\text{tr} \left( \frac{d}{dt} A \cdot \text{Adj}(A) \right) = \frac{d}{dt} \det A.$$

The proof is a nice exercise in linear algebra which we leave to the reader!

## 5. PROOF OF THE MAIN THEOREM, PART II

We have seen that the canonical map  $\text{Ad}: \text{Aut}(\mathbb{A}^n) \rightarrow \text{Aut}_{\text{Lie}}(\text{Vec}(\mathbb{A}^n))$  is an isomorphism (Theorem 3.1). It follows from Proposition 4.5 that every automorphism of  $\text{Vec}(\mathbb{A}^n)$  induces an automorphism of  $\text{Vec}^c(\mathbb{A}^n)$ . Moreover, since  $\text{Vec}^0(\mathbb{A}^n) = [\text{Vec}^c(\mathbb{A}^n), \text{Vec}^c(\mathbb{A}^n)]$  by Lemma 1.4, we get a canonical map  $\text{Aut}_{\text{Lie}}(\text{Vec}^c(\mathbb{A}^n)) \rightarrow \text{Aut}_{\text{Lie}}(\text{Vec}^0(\mathbb{A}^n))$  which is easily seen to be injective. Thus the main theorem from section 1 follows from the next result.

**Theorem 5.1.** *The canonical map  $\text{Ad}: \text{Aut}(\mathbb{A}^n) \rightarrow \text{Aut}_{\text{Lie}}(\text{Vec}^0(\mathbb{A}^n))$  is an isomorphism.*

The proof needs some preparation. The next proposition is a reformulation of some results from [Now86] and [LD12]. For the convenience of the reader we will give a short proof.

**Proposition 5.2.** *Let  $\xi_1, \dots, \xi_n \in \text{Vec}(\mathbb{A}^n)$  be pairwise commuting and  $K$ -linearly independent vector fields. Then the following are equivalent.*

- (i) *There is an étale morphism  $\eta: \mathbb{A}^n \rightarrow \mathbb{A}^n$  such that  $\eta^*(\partial_{x_i}) = \xi_i$  for all  $i$ ;*
- (ii)  *$\text{Vec}(\mathbb{A}^n) = \bigoplus_i K[x_1, \dots, x_n]\xi_i$ ;*
- (iii) *There exist  $f_1, \dots, f_n \in K[x_1, \dots, x_n]$  such that  $\xi_i(f_j) = \delta_{ij}$ ;*
- (iv)  *$\xi_1, \dots, \xi_n$  do not have a common DARBOUX polynomial.*

Recall that a common DARBOUX polynomial of the  $\xi_i$  is a non-constant  $f \in K[x_1, \dots, x_n]$  such that for all  $i$  we have  $\xi_i(f) = h_i f$  for some  $h_i \in K[x_1, \dots, x_n]$ .

*Proof.* (a) It follows from Remark 4.2 that (i) implies (ii) and (iii). Clearly, (ii) implies (iv) since a common DARBOUX polynomial for the  $\xi_i$  is also a common DARBOUX polynomial for the  $\partial_{x_i}$  which does not exist.

(b) We now show that (ii) implies (i), hence (iii), using the following well-known fact. If  $h_1, \dots, h_n \in K[x_1, \dots, x_n]$  satisfy the conditions  $\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$  for all  $i, j$ , then there is an  $f \in K[x_1, \dots, x_n]$  such that  $h_i = \frac{\partial f}{\partial x_i}$  for all  $i$ .

By (ii) we have  $\partial_{x_i} = \sum_k h_{ik} \xi_k$  for  $i = 1, \dots, n$ . The relations  $[\partial_{x_i}, \partial_{x_j}] = 0$  imply that  $\frac{\partial h_{ik}}{\partial x_j} = \frac{\partial h_{jk}}{\partial x_i}$  for all  $i, j, k$ , hence  $h_{ik} = \frac{\partial f_k}{\partial x_i}$  for suitable  $f_1, \dots, f_n \in K[x_1, \dots, x_n]$ . It is clear that the matrix  $(h_{ik})$  is invertible, hence the morphism  $\varphi := (f_1, \dots, f_n): \mathbb{A}^n \rightarrow \mathbb{A}^n$  is étale, and the claim follows from Remark 4.2.

(c) Assume that (iii) holds. Setting  $\xi_i = \sum_k h_{ik} \partial_{x_k}$  and applying both sides to  $f_j$ , we see that the matrix  $(h_{ik})$  is invertible, hence (ii). Thus the first three statements of the proposition are equivalent.

(d) Finally, assume that (iv) holds. Put  $\xi_i = \sum_k h_{ik} \partial_{x_k}$ . Since  $[\xi_i, \xi_j] = 0$  we get  $\xi_i(h_{jk}) = \xi_j(h_{ik})$  for all  $i, j, k$ . Now an easy calculation shows that  $\xi_k(\det(h_{ij})) = \text{Div}(\xi_k) \det(h_{ij})$ , and so  $\det(h_{ij}) \in K$ . If  $\det(h_{ij}) \neq 0$ , then (ii) follows.

If  $\det(h_{ij}) = 0$ , then  $\text{rank}(\sum_{i=1}^n K[x_1, \dots, x_n]\xi_i) = r < n$ , and we can assume that  $\text{rank}(\sum_{i=1}^r K[x_1, \dots, x_n]\xi_i) = r$ . Choose a non-trivial relation  $\sum_{i=1}^{r+1} f_i \xi_i = 0$  where  $\gcd(f_1, \dots, f_{r+1}) = 1$ . Since  $0 = \xi_j(\sum_{i=1}^{r+1} f_i \xi_i) = \sum_{i=1}^{r+1} \xi_j(f_i) \xi_i$  we see that  $\xi_j(f_i) \in K[x_1, \dots, x_n]f_i$ , and since the  $\xi_j$  are  $K$ -linearly independent at least one of the  $f_i$  is not a constant, contradicting (iv).  $\square$

The second main ingredient for the proof is the following result.

**Lemma 5.3.** *Let  $\xi_1, \xi_2 \in \text{Vec}^0(\mathbb{A}^n)$  be commuting vector fields. Assume that*

- (a)  $\xi_1$  and  $\xi_2$  have a common DARBOUX polynomial  $f$ ;
- (b) Each  $\xi_i$  acts locally nilpotently on  $\text{Vec}^0(\mathbb{A}^n)$ .

Then  $K[x_1, \dots, x_n]\xi_1 + K[x_1, \dots, x_n]\xi_2 \subset \text{Vec}(\mathbb{A}^n)$  is a submodule of rank  $\leq 1$ .

*Proof.* We will show that there are non-zero polynomials  $p_1, p_2$  such that  $p_1\xi_1 = p_2\xi_2$ . Set  $\xi_i(f) = h_i f$ . Using the formula  $\text{Div}(g\mu) = \mu(g) + g\text{Div}(\mu)$ , we see that  $\xi := h_1\xi_2 - h_2\xi_1 \in \text{Vec}^0(\mathbb{A}^n)$ . Moreover,  $\xi(f) = 0$ , and so  $f\xi \in \text{Vec}^0(\mathbb{A}^n)$ . Since

$$[\xi_1, \xi] = [\xi_1, h_1\xi_2] - [\xi_1, h_2\xi_1] = \xi_1(h_1)\xi_2 - \xi_1(h_2)\xi_1,$$

we get  $(\text{ad } \xi_1)^k \xi = \xi_1^k(h_1)\xi_2 - \xi_1^k(h_2)\xi_1$  and  $(\text{ad } \xi_1)^k(f\xi) = \xi_1^k(fh_1)\xi_2 - \xi_1^k(fh_2)\xi_1$ . Now, by assumption (b), there is a  $k > 0$  such that  $(\text{ad } \xi_1)^k \xi = (\text{ad } \xi_1)^k(f\xi) = 0$ , hence

$$\xi_1^k(h_1)\xi_2 = \xi_1^k(h_2)\xi_1 \text{ and } \xi_1^k(fh_1)\xi_2 = \xi_1^k(fh_2)\xi_1.$$

Thus the claim follows except if  $\xi_1^k(h_1) = \xi_1^k(h_2) = \xi_1^k(fh_1) = \xi_1^k(fh_2) = 0$ . We will show that this leads to a contradiction. Since  $\xi_1(f) = h_1 f$ , we get  $\xi_1^{k+1}(f) = 0$ . Now choose  $r, s \geq 0$  minimal with  $\xi_1^{r+1}(h_1) = 0$  and  $\xi_1^{s+1}(f) = 0$ . Then  $\xi_1^{r+s}(h_1 f) = \xi_1^r(h_1) \cdot \xi_1^s(f) \neq 0$ . On the other hand we have  $\xi_1^s(h_1 f) = \xi_1^{s+1}(f) = 0$ , a contradiction.  $\square$

Now we can prove the main result of this section.

*Proof of Theorem 5.1.* Let  $\theta$  be an automorphism of  $\text{Vec}^0(\mathbb{A}^n)$  as a Lie algebra, and put  $\xi_i := \theta(\partial_{x_i})$ . Then the vector fields  $\xi_1, \dots, \xi_n$  are commuting and  $K$ -linearly independent. Since every  $\partial_{x_i}$  acts locally nilpotently on  $\text{Vec}^0(\mathbb{A}^n)$  the same holds for each  $\xi_i$ .

We claim that the  $\xi_i$  do not have a common DARBOUX polynomial. Otherwise, Lemma 5.3 implies that  $\sum_i K[x_1, \dots, x_n]\xi_i \subset \text{Vec}(\mathbb{A}^n)$  has rank 1, i.e., there exist  $\xi \in \text{Vec}(\mathbb{A}^n)$ ,  $p_i \in K[x_1, \dots, x_n] \setminus K$  such that  $\xi_i = p_i \xi$ . We can assume that  $\xi$  is minimal, i.e., that  $\xi$  is not of the form  $p\xi'$  with a non-constant polynomial  $p$ .

For every  $\mu$  commuting with one of the  $\xi_i$ , we get  $0 = [\mu, \xi_i] = [\mu, p_i \xi] = \mu(p_i)\xi + p_i[\mu, \xi]$ , hence  $[\mu, \xi] \in K[x_1, \dots, x_n]\xi$ , because  $\xi$  is minimal. Since  $\text{Vec}^0(\mathbb{A}^n)$  is generated, as a Lie algebra, by elements commuting with one of the  $\partial_{x_i}$ , this implies that  $[\text{Vec}^0(\mathbb{A}^n), \xi] = [\theta(\text{Vec}^0(\mathbb{A}^n)), \xi] \subset K[x_1, \dots, x_n]\xi$ . But  $[\partial_{x_i}, \xi] \in K[x_1, \dots, x_n]\xi$  implies that  $[\partial_{x_i}, \xi] = 0$ , hence  $\xi = 0$ , a contradiction.

Now we use the implication (vi)  $\Rightarrow$  (iii) of Proposition 5.2 to see that there is an étale morphism  $\eta: \mathbb{A}^n \rightarrow \mathbb{A}^n$  such that  $\xi_i = \eta^*(\partial_{x_i})$  for all  $i$ . Similarly, there is an étale morphism  $\eta': \mathbb{A}^n \rightarrow \mathbb{A}^n$  such that  $\eta'^*(\partial_{x_i}) = \theta^{-1}(\partial_{x_i})$ . It follows that the composition  $\varphi := \eta' \circ \eta$  has the property that  $\varphi^*(\partial_{x_i}) = \partial_{x_i}$  for all  $i$ . Thus, by the following lemma,  $\varphi$  is a translation, hence  $\eta$  is an isomorphism. It follows that  $\text{Ad}(\eta) \circ \theta$  is the identity on  $\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$ , and so, again by the following lemma, we finally see that  $\theta = \text{Ad}(\psi)$  for some  $\psi \in \text{Aut}(\mathbb{A}^n)$ .  $\square$

**Lemma 5.4.** *Let  $\theta$  be an injective endomorphism of  $\text{Vec}^0(\mathbb{A}^n)$  such that  $\theta(\partial_{x_i}) = \partial_{x_i}$  for all  $i$ . Then  $\theta = \text{Ad}(\tau)$  where  $\tau: \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$  is a translation. In particular,  $\theta$  is an automorphism.*

*Proof.* We know that  $\sum_i K\partial_{x_i} = L(S)$  where  $S \subset \text{Aff}_n$  are the translations. Moreover,  $L(\text{Aff}_n)$  is the normalizer of  $L(S)$  in the Lie algebra  $\text{Vec}(\mathbb{A}^n)$ . Since  $\theta(L(S)) = L(S)$  we get  $\theta(L(\text{SAff}_n)) = L(\text{SAff}_n)$ , and so, by Lemma 3.2, there is an affine transformation  $g$  such that  $\text{Ad}(g)|_{L(\text{SAff}_n)} = \theta|_{L(\text{SAff}_n)}$ . Since  $\text{Ad}(g)$  is the



identity on  $L(S)$  we see that  $g$  is a translation. It follows that  $\text{Ad}(g^{-1}) \circ \theta$  is the identity on  $L(\text{SL}_n)$ , hence, by Lemma 3.3,  $\text{Ad}(g^{-1}) \circ \theta = \text{Ad}(\lambda E)$  for some  $\lambda \in K^*$ . But  $\lambda = 1$ , because  $\theta$  is the identity on  $L(S)$ .  $\square$

## REFERENCES

- [Bav13] V. V. Bavula, *The group of automorphisms of the Lie algebra of derivations of a polynomial algebra*, preprint (2013), arXiv:1304.3836 [math.RA].
- [BKY12] Alexei Belov-Kanel and Jie-Tai Yu, *Lifting of the automorphism group of polynomial algebras*, arXiv:1207.2045v1 (2012).
- [Kra11] Hanspeter Kraft, *Algebraic transformation groups: An introduction*, Mathematisches Institut, Universität Basel, <http://www.math.unibas.ch/kraft>, 2011.
- [KZ13] Hanspeter Kraft and Mikhail Zaidenberg, *Locally finite group actions and vector fields*, Preprint, 2013.
- [Kul92] Vik. S. Kulikov, *Generalized and local Jacobian problems*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 5, 1086–1103.
- [Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
- [LD12] Jiantao Li and Xiankun Du, *Pairwise commuting derivations of polynomial rings*, Linear Algebra Appl. **436** (2012), no. 7, 2375–2379.
- [Now86] Andrzej Nowicki, *Commutative bases of derivations in polynomial and power series rings*, J. Pure Appl. Algebra **40** (1986), no. 3, 275–279.
- [Pro07] Claudio Procesi, *Lie groups*, Universitext, Springer, New York, 2007, An approach through invariants and representations.
- [Reg13] Andriy Regeta, *Lie subalgebras of vector fields and the Jacobian conjecture*, arXiv:1311.0232 [math.RA], 2013.
- [Sha66] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) **25** (1966), no. 1-2, 208–212.
- [Sha81] ———, *On some infinite-dimensional groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 214–226, 240.

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